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Lie group formalism for difference equations

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Abstract. The methods of Lie group analysis of differential equations are generalized so as to provide an infinitesimal formalism for calculating symmetries of difference equations. Several examples are analysed, one of them being a nonlinear difference equation. For the linear equations the symmetry algebra of the discrete equation is found to be isomorphic to that of its continuous limit.

Résumé. Les méthodes de la théorie de groupes utilisées pour traiter les équations différentielles sont généralisées au cas des équations aux différences finies. Plusieurs examples sont analysés, parmi ceux-ci celui d'une équation non linéaire. Dans tous les cas linéaires il s'avère que l'algèbre de symétrie de l'équation discrète est isomorphe à celle de sa limite continue.

1. Introduction

The purpose of this paper is to develop an algebraic formalism for calculating symmetries of difference equations. The equations to be studied will involve functions $u(x)$, where the dependent variables $u \in \mathbb{R}^p$ and independent ones $x \in \mathbb{R}^q$ are considered to be continuous. The equations themselves will be discrete, i.e. written on a uniform lattice with spacings $\sigma_i > 0$ (in the direction x_i). Thus, instead of involving partial derivatives of $u(x)$, the equations will contain finite differences (variations) of the form

$$\Delta_{x_i} u = \frac{1}{\sigma_i}\{u(x_1, \ldots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \ldots, x_q) - u(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_q)\} \quad (1.1)$$

and also shift operators T_{x_i}

$$T_{x_i} u = u(x_1, \ldots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \ldots, x_q)\,. \quad (1.2)$$

The spacings, variations and shift operators are related:

$$\Delta_{x_i} = \frac{1}{\sigma_i}(T_{x_i} - 1)\,. \quad (1.3)$$

The symmetries that we will introduce will constitute a 'minimal' extension of Lie point symmetries of differential equations. For difference equations we will consider

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transformations that take solutions into solutions. They are not strictly point transformations, in that the transformed function $\widetilde{u}(x)$ depends not only on the old u and x, but also on shifted values $T_{x_i}u(x)$. In the continuous limit with $\sigma_i \to 0$, $T_{x_i} \to 1$, $\Delta_{x_i} \to \partial_{x_i}$, the transformations will reduce to point ones.

Our aim is to provide group-theoretical tools that are equally efficient for solving difference equations, as their continuous limits are in the case of differential equations [1–5]. We recall here that once the symmetry group of a differential equation is found, it can be used for many purposes. It will transform given solutions into new ones, often trivial solutions into interesting ones. The symmetry group can be used to perform symmetry reduction, i.e. reduce the order of an ordinary differential equation, or the number of independent variables in a partial differential equation. Isomorphies between symmetry groups of differential equations can be used to identify equivalent equations, in particular to determine whether a nonlinear equation is linearizable by a point transformation [2]. Symmetry groups can also serve as indicators of integrability [6–8] by Lax pair techniques.

The motivation for such a programme comes from the fact that difference equations are becoming increasingly important in physical and other applications. They occur naturally whenever discrete phenomena are studied. Obvious examples are spin lattices in statistical mechanics, phenomena in crystals, molecular chains [9], etc. Difference equations and their q-analogues play a crucial role in the representation theory of quantum groups and hence in special function theory [10–13]. Difference equations also occur in studies of essentially continuous phenomena, when differential equations are approximated by discrete ones, for instance in numerical studies.

Several different approaches to symmetries of discrete equations exist in the literature.

In the first of them [14–18] one starts out from a given difference, or differential difference equation for a function $u \in \mathbb{R}^p$, depending on a set of continuous variables $x \in \mathbb{R}^q$ and *discrete variables* $n = (n_1, \ldots, n_r)$. The equation involves derivatives in the continuous independent variables x and shifts in the discrete variables n. Thus, in the equation, we have entries like $u(x, n+k)$, $u_{x_i}(x, n+k_i)$, etc, with (the discrete vectors) k and k_i in some finite range. The method provides symmetry transformations, generated by vector fields of the form

$$X = \sum_i \xi_i(x, u)\partial_{x_i} + \sum_\alpha \phi_\alpha(n, x, u)\partial_{u_\alpha} \tag{1.4}$$

where $u = u(x, n)$. The method for finding the vector fields X, i.e. the coefficients ξ_i and ϕ_α, involves the construction of the prolongation $\mathrm{pr}^N X$ of X (N is the order of the highest derivative in the equation). The prolongation acts on functions of x, u and the derivatives of u, evaluated at all the points $n + k$ where the functions $u(n, x)$ and their derivatives are given [15, 16]. The determining equations for the coefficients ξ_i and ϕ_α are obtained by requiring that the prolongation $\mathrm{pr}^N X$ should annihilate the equation on its solution set. The algorithm thus coincides with the one used for differential equations [1–5], the only difference being in the way in which the vector field is prolonged. Once the continuous symmetries are found, the discrete ones are obtained separately, essentially by inspection (e.g. discrete translations, or rotations, acting on the variables n).

The described method is quite general, equally applicable to linear and nonlinear equations. The disadvantage is that when passing from continuous to discrete equations in this manner, some of the symmetries are lost. Indeed, since the discrete variables n are required to have integer values, it is difficult to envisage, for example, dilations of such variables.

A second method was introduced recently for finding symmetries of linear difference equations [19, 20]. All variables in the problem are assumed to be continuous, with their increments discrete, as in (1.1)–(1.3). The equations are written in the form

$$Lu(x) = 0 \qquad u \in \mathbb{R}^p \quad x \in \mathbb{R}^q \tag{1.5}$$

where L is a polynomial in the finite differences Δ_{x_i} with coefficients that can be functions of x and T_{x_i}. The symmetry operators in this approach have the form

$$\widehat{X} = \sum_{i=1}^{q} \xi_i(x, T_x)\Delta_{x_i} + f(x, T_x) \tag{1.6}$$

i.e. they are themselves finite difference operators.

The symmetry operators are required to commute with the operator L on the solution set of (1.5)

$$[L, \widehat{X}] = \lambda L \tag{1.7}$$

where λ can be a constant, a function of x, or a lower-order difference operator. Equations of the type (1.7) for difference operators are actually hard to solve. The approach actually adopted in [19, 20] is to find operators $\widehat{X}$ such that $w = \widehat{X}u$ is a solution of (1.5), whenever u is one. This requires knowing a complete set of solutions and implies that these operators satisfy equations of the form (1.7). Upon investigating the commutation relations of the symmetry generators, it was found, in all the examples studied, that they realize a Lie algebra isomorphic to the one obtained in the continuous limit. These symmetries were further used to obtain solutions involving discrete analogues of special functions.

In spirit, this method is a straightforward adaptation of symmetry methods used by physicists to study symmetries and degeneracy problems in quantum mechanics. Below we shall call it 'the method of linear operators'. The big advantage of this method is that in all the cases studied to date it provides symmetry algebras for discrete equations that are isomorphic to those of their continuous limits.

Their concrete realizations are of course different. The disadvantage of the method is that it is only applicable to linear equations. Moreover, as we shall see below, even for linear equations the method of linear operators may not give all symmetries (either in the continuous, or in the discrete case).

A third approach to symmetries of discrete equations is due to Dorodnitsyn and collaborators [21, 22]. In this case the aim is to discretize a differential equation in such a manner as to preserve all Lie point symmetries of the continuous equation. The symmetry operators in this approach remain in their original (continuous) form. The lattice is introduced in a very specific way, so as to be compatible with the original symmetries. The group transformations then act not only on the independent and dependent variables, but also on the lattice. For other results on symmetries of discrete equations, see e.g. [23, 24].

The purpose of this paper, already stated at the beginning of this introduction, can be reformulated as follows. We wish to combine the advantages of the first two approaches described above. That is, to develop a prolongation formalism for 'discrete vector fields', applicable to linear and nonlinear equations, providing Lie symmetries in an algorithmic manner. Moreover, in the continuous limit we should recover all Lie point symmetries of the corresponding differential equations.

Section 2 of this paper is devoted to a general formalism for obtaining symmetries of difference equations. We first recapitulate some results concerning the formalism of evolutionary vector fields for differential equations. The general prolongation formalism for difference equations is developed in section 2.2. Section 3 is devoted to examples. We treat the discrete heat equation with a general 'potential' and then specify the potential to specific cases with interesting symmetry algebras. We also obtain the generators of both linear and nonlinear symmetry transformations for the linear equation $\Delta_{xx}u(x) = 0$. A nonlinear difference equation is analysed in section 3.4. Some conclusions are presented in section 4.

2. Prolongation formalism for difference equations

2.1. Evolutionary vector fields for differential equations

Let us first consider a differential equation

$$E(x, u_{x_i}, u_{x_i x_k}, \ldots) = 0 \tag{2.1}$$

where E is some sufficiently smooth scalar function of the independent variables $x \in \mathbb{R}^q$, the dependent variable u (for simplicity of notation a scalar) and its derivatives $u_{x_i, u_{x_i x_k}}, \ldots,$ up to order N.

The Lie algebra L of the symmetry group G of local point transformations can be realized in terms of vector fields of the form

$$X = \sum_{i=1}^{q} \xi_i(x, u)\partial_{x_i} + \phi(x, u)\partial_u\,. \tag{2.2}$$

The Nth prolongation of X must annihilate the equation on its solution set:

$$\mathrm{pr}^N X\ E|_{E=0} = 0\,. \tag{2.3}$$

The prolongation is constructed in a standard manner [1–5], that has been implemented in computer programs in many symbolic languages (see e.g. [25]).

Equation (2.3) provides a set of linear partial differential equations for the functions $\xi_i(x, u)$ and $\phi(x, u)$. Once they are solved, we have the symmetry algebra L, with X either depending on a finite number r of integration constants, or on one or more arbitrary functions. In the first case we have $\dim L = r$, in the second the algebra is infinite dimensional.

Alternatively [1], we can use the formalism of evolutionary vector fields

$$X_e = Q\partial_u \qquad Q = \xi_i(x, u)u_{x_i} - \phi(x, u) \tag{2.4}$$

where ξ_i and ϕ have the same meaning as in (2.2). The prolongation formulae in the evolutionary formalism are quite simple, namely

$$\begin{aligned} &\mathrm{pr}^N X_e = Q\partial_u + Q^{x_i}\partial_{u_{x_i}} + Q^{x_i x_j}\partial_{u_{x_i x_j}} + \cdots \\ &Q^{x_i} = \mathrm{D}_{x_i} Q \qquad Q^{x_i x_j} = \mathrm{D}_{x_i}\mathrm{D}_{x_j} Q, \ldots, \end{aligned} \tag{2.5}$$

where D_{x_i} is the total derivative with respect to x_i.

The evolutionary formalism is the one that we shall adapt to difference equations, so let us present some further results for it.

The determining equations for the symmetry operator X_e, i.e. for the functions ξ_i and ϕ are obtained by requiring that we have

$$\mathrm{pr}\, X_e E|_{E=0} = 0\,. \tag{2.6}$$

The commutation relations in the Lie algebra are obtained by commuting the first prolongations of the corresponding vector fields and then projecting the results onto the algebra L:

$$\begin{aligned}[X_{1e}, X_{2e}] &= [\mathrm{pr}^{(1)} X_{1e}, \mathrm{pr}^{(1)} X_{2e}]|_L \\ &= \left\{ Q_1 \frac{\partial}{\partial u} Q_2 - Q_2 \frac{\partial}{\partial u} Q_1 + Q_1^{x_i} \frac{\partial}{\partial u_{x_i}} Q_2 - Q_2^{x_i} \frac{\partial}{\partial u_{x_i}} Q_1 \right\} \partial_u \end{aligned} \tag{2.7}$$

(the $\partial_{u_{x_i}}$ terms are cut off by the projection onto the algebra L).

The symmetry transformations are obtained from the evolutionary fields by integrating them, together with the original equation. Thus a one-parameter group of symmetry transformations is obtained by solving the system

$$\frac{\partial \widetilde{u}(x, \lambda)}{\partial \lambda} = Q(x, \widetilde{u}(x), \widetilde{u}_{x_i}(x)) \qquad \widetilde{u}(x, \lambda)|_{\lambda=0} = u(x) \tag{2.8}$$

(possibly together with (2.1)). The result of the integration can be represented formally as

$$\mathrm{e}^{\lambda X_e} u(x) = \widetilde{u}(x, \lambda) \equiv u(x) + \lambda Q(x, \tilde{u}(x), \tilde{u}_{x_i}(x)) + \mathcal{O}(\lambda^2)\,. \tag{2.9}$$

The formalism of commuting linear operators is related to the Lie prolongation formalism in a simple manner. Indeed, let us restrict (2.1) to be linear and write it in the form (1.5), where L is a linear differential operator. The local group transformation corresponding to the Lie algebra element (2.2) is given by

$$\widetilde{u}(\widetilde{x}) = (\mathrm{e}^{-\lambda X} u)(x) \qquad \widetilde{x} = \mathrm{e}^{\lambda X} x\,. \tag{2.10}$$

Let us now further restrict the coefficients ξ_i and ϕ in (2.2) to have the form

$$\xi_i = \xi_i(x) \qquad \phi = -f(x)u \tag{2.11}$$

and expand $\widetilde{u}(\widetilde{x})$ and $\widetilde{x}$ in (2.10) into power series, keeping only the terms propositional to 1 and λ. We find that $Lu = 0$ implies

$$\widehat{X} = \xi_i(x)\partial_{x_i} + f(x)\,. \tag{2.12}$$

Conversely, if we require that the operator $\widehat{X}$ of (2.12) should satisfy the commutation relation (1.7), with L the corresponding linear differential operator and λ a function of x, or a differential operator, then $\widehat{X}u$, $\widehat{X}^p u$ $(p \in \mathbb{Z}^{\geqslant})$ and $\mathrm{e}^{\lambda \widehat{X}} u$ are all solutions, if $u(x)$ is one.

Thus, for linear equations, instead of using the Lie formalism, we can simply look for operators $\widehat{X}$ of the form (2.12), satisfying the commutation relation (1.7). The operator

$$X_e = (\widehat{X} u)\partial_u \tag{2.13}$$

will then be a Lie symmetry operator in evolutionary form.

It should, however, be stressed that this linear formalism may only provide a subalgebra of the Lie symmetry algebra, since the restriction (2.11), i.e. ξ_i independent of u, ϕ proportional to u, may result in a loss of some symmetries (see the second example in section 3 below).

2.2. Evolutionary formalism for difference equations

Let us write a general difference equation, involving one scalar function $u(x)$ of p independent variables $x = (x_1, \ldots x_p)$ evaluated at a finite number of points on a lattice. Symbolically we write it as

$$E(x, T^{\alpha} u(x), T^{\beta_i} \Delta_{x_i} u(x), T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \ldots) = 0 \tag{2.14}$$

where E is some given function of its arguments and we have put, for example,

$$T^{\alpha}u(x) = T_{x_1}^{\alpha_1} T_{x_2}^{\alpha_2} \dots T_{x_p}^{\alpha_p} u(x) \qquad a_i \leqslant \alpha_i \leqslant b_i \qquad i = 1, \dots, p \tag{2.15}$$

where α_i are integers running through some fixed finite range (i.e. a_i and b_i are some fixed integers). The shift operators T^{β_i}, $T^{\gamma_{ij}}, \dots$ are defined similarly and the corresponding integers $\beta_i \sim (\beta_{i1}, \dots, \beta_{ip})$, $\gamma_{ij} \sim (\gamma_{ij1}, \dots, \gamma_{ijp}), \dots,$ also vary over some finite range.

The function E may also depend explicitly on the spacings σ_i, however, σ_i, T_{x_i} and Δ_{x_i} are not independent, since we have (1.3). Thus, to avoid redundances in the equations and in the vector fields, we shall not allow expressions of the type $\sigma_i^a \Delta_{x_i}$ with $a \in \mathbb{Z}^{>}$. They can always be eliminated using (1.3). Negative powers of σ_i are also disallowed. They can either be absorbed in the variations $\Delta_{x_i} u$, or they can have a singular continuous limit.

Let us now consider a 'discrete vector field' in evolutionary form. We shall postulate that it can be written as

$$X_e = Q\partial_u \qquad Q = \sum_i \xi_i(x, T^a u) T^b \Delta_{x_i} u - \phi(x, T^c u) \tag{2.16}$$

where T^a, T^b and T^c have the same meaning as in (2.14). Notice that in (2.16) the variations $\Delta_{x_i} u$ enter linearly, whereas the dependence on u (shifted, or not) is arbitrary. In the continuous limit (2.16) reduces to (2.4).

The essential task is to present the consistent prolongation of the vector field (2.16). The group transformations, generated by X_e will take functions $u = u(x)$ into transformed functions $\widetilde{u} = \widetilde{u}(x)$. The group transformations generated by the prolongation $\mathrm{pr}^N X_e$ must also transform the variations $\Delta_{x_i} u$, $\Delta_{x_i} \Delta_{x_j} u$ (up to order N) into the variations of $\widetilde{u}$ with respect to x_i, and to do this at all points of the lattice.

This also means that $\mathrm{pr}^N X_e$ will act on functions of x, u and the variations at different points of the lattice, i.e. functions of the form of the considered equation (2.14).

The prolongation formula satisfying the above requirements has the form

$$\mathrm{pr}^N X_e = \left\{ \sum_{\alpha} T^{\alpha} Q \partial_{T^{\alpha} u} + \sum_{\beta_i} T^{\beta_i} Q^{x_i} \partial_{T^{\beta_i} \Delta_{x_i} u} + \sum_{\gamma_{ij}} T^{\gamma_{ij}} Q^{x_i x_j} \partial_{T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u} + \dots \right\}. \tag{2.17}$$

The shift operators $T^{\alpha}, \dots,$ were defined in (2.15), the summations are over all values (all sites) represented in (2.14). The coefficients Q^{x_i}, $Q^{x_i x_j}, \dots$ are total variations of the coefficient Q in the discrete evolutionary field (2.16). We have

$$Q^{x_i} = \Delta_{x_i}^T Q \qquad Q^{x_i x_j} = \Delta_{x_i}^T \Delta_{x_j}^T Q \tag{2.18}$$

and similarly for higher-order prolongations.

The total variation Δ_x^T acts on functions of $x, u, \Delta_x u, \dots$ according to the following definition:

$$\begin{aligned} \Delta_x^T f(x, u(x), \Delta_x u(x), \dots) = \frac{1}{\sigma} [& f(x + \sigma, u(x + \sigma), (\Delta_x u)(x + \sigma), \dots) \\ & - f(x, u(x), (\Delta_x u)(x), \dots)] \end{aligned} \tag{2.19}$$

(for each $x = x_i$). The action of the 'partial' variation Δ_x, on the other hand, is

$$\begin{aligned} &\Delta_x f(x, u(x), \Delta_x u(x), \dots) \\ &\quad = \frac{1}{\sigma} [f((x + \sigma), u(x), \Delta_x u(x), \dots) - f(x, u(x), \Delta_x u(x), \dots)] \end{aligned} \tag{2.20}$$

for each independent variable x_i.

The main difference between the prolongation formula (2.17), and the corresponding one in the continuous case, i.e. equation (2.5), is precisely the summation over different sites on the lattice. In the continuous limit $\sigma_i \to 0$ we have $T \to 1$ for all shift operators and each summation contracts to a single term.

Once constructed, the discrete vector fields in their evolutionary form (2.16) and their prolongation (2.17) can be treated in the same manner as in the continuous case. Commutators are calculated as in (2.7), i.e. by commuting the first prolongations and projecting onto L. Symmetry transformations are obtained by solving equation (2.8). This may, however, be difficult, since (2.8) is a first-order differential difference equation involving differentiation in λ, but discrete derivatives (variations) in the p variables x_i. In the continuous case it is a first-order linear PDE that can be solved by the method of characteristics.

Luckily, in most applications one deals with the Lie algebra, rather than the Lie group. The explicit group transformations are rarely needed.

The relation between the prolongation formalism, and the linear operator one, is the same as in the continuous case. Indeed, consider the special case when equation (2.14) is linear. The operator $\widehat{X}$ of (1.6), satisfying (1.7), can be obtained by restricting Q to satisfy

$$Q = [\xi_i(x) T_x^{\beta_i}(\Delta_{x_i} u) + f(x, T_x)u] = \widehat{X} u\,. \tag{2.21}$$

The algorithm for calculating the symmetry algebra is given by equation (2.6), i.e. it is the same as in the continuous case. The determining equations for the functions ξ_i and ϕ are read off as coefficients of linearly independent expressions in the (possibly shifted) discrete derivatives $(T^\alpha \Delta_{x_i} u)$, $(T^\beta \Delta_{x_i x_i} u), \ldots$.

It should be pointed out, however, that technical difficulties occur, that make the general formalism more difficult to use, than in the continuous case.

The main problem is inherent in the total variations $\Delta_x^T f$, defined in (2.19). Indeed, total derivatives are expressed in a simple manner in terms of partial derivatives, e.g.

$$\mathrm{D}_x f(x, u, u_x, \ldots) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} u_x + \frac{\partial f}{\partial u_x} u_{xx} + \cdots.$$

As yet, no such formula is available for total variations. To deal with them we can either make use of a Taylor expansion of the function on the right-hand side of (2.19), or introduce a mechanism for expressing the total variations in terms of partial ones. The first possibility introduces derivatives of all orders, the second one will introduce further difference equations that are not easy to solve. An additional obvious problem is the presence of *a priori* arbitrary functions of the shift operators T_{x_i}.

These technical difficulties really only arise for nonlinear equations.

3. Examples

Let us now apply the formalism of section 2 to some specific equations, three linear and one nonlinear one.

3.1. Discrete heat equation

The equation that we consider is

$$[\Delta_t - \Delta_{xx} + g(x, t, T_x, T_t)]u(x, t) = 0 \tag{3.1}$$

where g is some given function of x, t and the two shift operators T_x and T_t.

In agreement with section 2, equation (2.16), we write the evolutionary vector field in the form

$$X_e = [\tau \Delta_t u + \xi \Delta_x u + f u]\partial_u \equiv Q\partial_u \tag{3.2}$$

where τ, ξ and f are functions of x, t, T_x and T_t. They can also depend explicitly on the x and t spacings, σ_x and σ_t (to some non-negative powers). We apply the second prolongation $\mathrm{pr}^2 X_e$ to (3.1) and require that the result should vanish on the solution set:

$$Q^t - Q^{xx} + gQ|_{\Delta_{xx}u=\Delta_t u+gu} = 0\,. \tag{3.3}$$

The coefficients $Q^t = \Delta_t^T Q$ and $Q^{xx} = \Delta_{xx}^T Q$ depend explicitly on u and the variations of u, up to third order. We use (3.1) and its difference consequences to eliminate $\Delta_{xx}u$, $\Delta_{xxx}u$ and $\Delta_{xxt}u$. All $\Delta_{tt}u$ terms cancel and we are left, in equation (3.3), with the expressions $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, all figuring linearly only. The coefficient of each of these expressions must vanish and we obtain four determining equations for ξ, τ, and f in (3.2). They are:

$$\Delta_x \tau = 0 \tag{3.4}$$

$$-(\Delta_t\tau)T_t + 2(\Delta_x\xi)T_x + [\tau, g] = 0 \tag{3.5}$$

$$-(\Delta_t\xi)T_t + (\Delta_{xx}\xi)T_x^2 + 2(\Delta_x f)T_x + [\xi, g] = 0 \tag{3.6}$$

$$-(\Delta_t f)T_t + (\Delta_{xx} f)T_x^2 + 2(\Delta_x\xi)T_x g + \xi(\Delta_x g)T_x + \tau(\Delta_t g)T_t + [f, g] = 0\,. \tag{3.7}$$

We shall not analyse the symmetries of (3.1) for all possible functions g, though in principle that would be possible (though not easy even in the continuous case). Instead, we consider some special cases.

3.1.1. The 'free' heat equation: $g(x, t, T_x, T_t) = 0$. The determining equations (3.4)–(3.7) are easily solved to yield

$$\tau = t^{(2)}\tau_2 + t\tau_1 + \tau_0 \tag{3.8}$$

$$\xi = \tfrac{1}{2}x(\tau_1 + 2t\tau_2)T_t T_x^{-1} + t\xi_1 + \xi_0 \tag{3.9}$$

$$f = \tfrac{1}{4}x^{(2)}\tau_2 T_t^2 T_x^{-2} + \tfrac{1}{2}t\tau_2 T_t + \tfrac{1}{2}x\xi_1 T_t T_x^{-1} + \gamma \tag{3.10}$$

where τ_0, τ_1, τ_2, ξ_0, ξ_1 and γ are arbitrary functions of the shift operators T_x and T_t and also of the spacings σ_x and σ_t.

In equations (3.8) and (3.10) we have introduced factorial functions, or Pochhammer symbols $x^{(n)}$ (or $t^{(n)}$) satisfying

$$\begin{aligned}
&\Delta_x x^{(n)} = nx^{(n-1)} \qquad n \in \mathbb{Z}^{>}\\
&x^{(n)} = x(x-\sigma_x)\ldots(x-(n-1)\sigma_x)\\
&x^{(0)} = 1\\
&\Delta_x x^{(-n)} = -nx^{(-n-1)} \qquad n \in \mathbb{Z}^{>}
\end{aligned} \tag{3.11}$$

$$x^{(-n)} = \frac{1}{x(x+\sigma_x)\ldots(x+(n-1)\sigma_x)} \tag{3.12}$$

(and similarly for $t^{(n)}$).

At first glance, each of the functions τ_i, ξ_a and γ yields an infinite-dimensional subalgebra of the symmetry algebra. We can, however, pick out a six-dimensional algebra, isomorphic to the symmetry algebra of the continuous heat equation. Indeed, let us construct a basis, corresponding to six different choices of the coefficients in (3.8)–(3.10), respectively (in each case all unspecified coefficients are chosen to vanish):

$$
\begin{aligned}
&\tau_0 = 1: \qquad P_0 = (\Delta_t u)\partial_u \\
&\xi_0 = 1: \qquad P_1 = (\Delta_x u)\partial_u \\
&\gamma = 1: \qquad W = u\partial_u \\
&\xi_1 = 2T_t^{-1} \qquad \gamma = \tfrac{1}{2}\sigma_x T_x^{-1} \\
&B = (2tT_t^{-1}\Delta_x u + xT_x^{-1}u + \tfrac{1}{2}\sigma_x T_x^{-1}u)\partial_u \\
&\tau_1 = 2T_t^{-1} \qquad \gamma = 1 - \tfrac{1}{2}T_x^{-1} \\
&D = [2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + (1 - \tfrac{1}{2}T_x^{-1})u]\partial_u \\
&\tau_2 = T_t^{-2} \qquad \xi_1 = \sigma_x T_x^{-1}T_t^{-1} \qquad \gamma = -\tfrac{1}{16}\sigma_x^2 T_x^{-2} \\
&K = [t^2T_t^{-2}\Delta_t u + txT_t^{-1}T_x^{-1}\Delta_x u + \tfrac{1}{4}x^2T_x^{-2}u + t(T_t^{-2} - \tfrac{1}{2}T_t^{-1}T_x^{-1})u - \tfrac{1}{16}\sigma_x^2T_x^{-2}u]\partial_u \,.
\end{aligned}
\tag{3.13}
$$

The shift operators, T_x and T_t, themselves represent discrete symmetries of (3.1), i.e. $T_x^n T_t^m u$ is a solution, if u is one. Hence the general element (3.2) of the symmetry algebra, with τ, ξ and f as in (3.8)–(3.10) can be viewed as a linear combination of elements of the algebra (3.13), with coefficients that are polynomials in T_x and T_t.

The symmetry algorithm thus provides not only the symmetry algebra itself, but also elements that properly belong to an analogy of the enveloping algebra in the continuous case.

3.1.2. Nonzero potential $g(x, t, T_x, T_t) \neq 0$. For the sake of simplicity, let us first of all restrict the potential to the form

$$g = g(x, T_x)\,. \tag{3.14}$$

We also make some restrictions on the dependence of the symmetry generators on the shift operators. First of all, we impose $\tau = \tau(t, T_t)$ (no T_x dependence). We then solve (3.4)–(3.6) to obtain

$$\tau = \tau(t, T_t) \tag{3.15}$$

$$\xi = \tfrac{1}{2}x(\Delta_t\tau)T_tT_x^{-1} + \alpha(t, T_t) \tag{3.16}$$

$$f = \tfrac{1}{8}x^{(2)}(\Delta_{tt}\tau)T_t^2T_x^{-2} + \tfrac{1}{2}x(\Delta_t\alpha)T_tT_x^{-1} - \tfrac{1}{4}\Delta_x^{-1}[x(\Delta_t\tau)T_tT_x^{-1}, g]T_x^{-1} + \beta(t, T_t)\,. \tag{3.17}$$

The fact that α and β in (3.15)–(3.17) do not depend on T_x is again a restriction on the symmetries considered.

The remaining determining equation (3.7) relates the functions $g(x, T_x)$, $\tau(t, T_t)$, $\alpha(t, T_t)$ and $\beta(t, T_t)$. It simplifies greatly if the commutator $[f, g]$ vanishes.

Two cases of interest when this happens are

$$g = cx^{(p)}T_x^{-p} \qquad p \in \mathbb{Z}^{\geqslant} \tag{3.18}$$

$$g = cT_x x^{(-p)}T_x^{p-1} \qquad p \in Z^{>}\,. \tag{3.19}$$

For both these potentials we have

$$[xT_x^{-1}, g] = 0 \qquad [x^{(2)}T_x^{-2}, g] = 0\,. \tag{3.20}$$

Hence f of equation (3.17) simplifies to

$$f = \tfrac{1}{8}x^{(2)}(\Delta_{tt}\tau)T_t^2T_x^{-2} + \tfrac{1}{2}x(\Delta_t\alpha)T_tT_x^{-1} + \beta(t, T_t)\,. \tag{3.21}$$

Moreover, equation (3.7) can now be rewritten as

$$-\tfrac{1}{8}x^{(2)}(\Delta_{ttt}\tau)T_t^3T_x^{-2} - \tfrac{1}{2}x(\Delta_{tt}\alpha)T_t^2T_x^{-1} - (\Delta_t\beta)T_t + \tfrac{1}{4}(\Delta_{tt}\tau)T_t^2 + (\Delta_t\tau)T_t g$$
$$+\tfrac{1}{2}x(\Delta_t\tau)T_tT_x^{-1}(\Delta_x g)T_x + \alpha(\Delta_x g)T_x = 0\,. \qquad (3.22)$$

Equation (3.22) holds for any potential g satisfying $[f, g] = 0$, in particular, for the potentials (3.18) and (3.19).

We shall now restrict $g(x, T_x)$ even further, namely to two cases that have interesting symmetry algebras in the continuous case when we have $\sigma_x \to 0$, $\sigma_t \to 0$, $T_x \to 1$, $T_t \to 1$. The continuous heat equation has a six-dimensional symmetry algebra, isomorphic to that of the free equation for $g = cx^2$. It has a four-dimensional symmetry algebra that is a subalgebra of that of the free equation, for $g = ax^{-2}$ (see [26, 27] for similar results for the Schrödinger equation).

Let us consider the discrete analogues of these two cases.

3.1.3. *(i) Discrete harmonic oscillator.*

$$g(x, T_x) = k^2x^{(2)}T_x^{-2} \qquad k \in \mathbb{R}^{>}\,. \qquad (3.23)$$

In this case equation (3.22) has terms proportional to $x^{(2)}$, x and x^0. Their coefficients must vanish separately, since τ, α and β do not depend on x. The corresponding equations are

$$\begin{aligned}(\Delta_{ttt}\tau) &= 16k^2(\Delta_t\tau)T_t^{-2}\\ \Delta_{tt}\alpha &= 4k^2\alpha T_t^{-2}\\ \Delta_t\beta &= \tfrac{1}{4}(\Delta_{tt}\tau)T_t\,.\end{aligned} \qquad (3.24)$$

To solve these equations, we introduce a 'shifted discrete exponential' $E_m(t)$ satisfying

$$\Delta_t E_m(t) = mE_m(t)T_t^{-1}\,. \qquad (3.25)$$

The solution of (3.25) can be written as

$$E_m(t) = (1 + mT_t^{-1}\sigma_t)^{t/\sigma_t}\,. \qquad (3.26)$$

In terms of this function we have the general solution of (3.24) as

$$\begin{aligned}\tau &= E_{4k}(t)\tau_1 + E_{-4k}(t)\tau_2 + \tau_0\\ \alpha &= E_{2k}(t)\alpha_1 + E_{-2k}(t)\alpha_2\\ \beta &= k(E_{4k}(t)\tau_1 - E_{-4k}(t)\tau_2) + \gamma\end{aligned} \qquad (3.27)$$

where τ_1, τ_2, τ_0, α_1, α_2 and γ are functions of T_t (and the spacings σ_x, σ_t).

The corresponding evolutionary vector fields are

$$\begin{aligned}P_0 &= \tau_0\Delta_t u\partial_u \qquad L = \gamma u\partial_u\\ T_1 &= E_{4k}(t)\tau_1[\Delta_t u + 2kxT_x^{-1}\Delta_x u + 2k^2x^{(2)}T_x^{-2}u + ku]\partial_u\\ T_2 &= E_{-4k}(t)\tau_2[\Delta_t u - 2kxT_x^{-1}\Delta_x u + 2k^2x^{(2)}T_x^{-2}u - ku]\partial_u\\ A_1 &= E_{2k}(t)\alpha_1(\Delta_x u + kxT_x^{-1}u)\partial_u\\ A_2 &= E_{-2k}(t)\alpha_2(\Delta_x u - kxT_x^{-1}u)\partial_u\,.\end{aligned} \qquad (3.28)$$

As in the case of $g = 0$ it is possible to fix the functions τ_i, α_i and γ and choose a basis, so as to obtain a six-dimensional Lie algebra isomorphic to that of the free heat equation (the discrete and the continuous one). We shall not go into that here.

3.1.4. (ii) Discrete 'centrifugal barrier'.

$$g(x, T_x) = T_x x^{(-2)} T_x \,. \tag{3.29}$$

Equation (3.22) now contains terms of the form $x^{(2)}$, x, x^0 and $x^{(-2)}$. We obtain four determining equations. Solving them we find τ, ξ and f in the general element of the symmetry algebra to be

$$\begin{aligned} \tau &= \tau_0 + t\tau_1 + t^{(2)}\tau_2 \\ \xi &= \tfrac{1}{2} x(\tau_1 + 2t\tau_2) T_t T_x^{-1} \\ f &= \tfrac{1}{4} x^{(2)} \tau_2 T_t^2 T_x^{-2} + \tfrac{1}{2} t\tau_2 T_t + \gamma \end{aligned} \tag{3.30}$$

where τ_i and γ are functions of T_t, σ_t and σ_x.

By fixing τ_i and γ appropriately, we can select a four-dimensional symmetry algebra from the vector fields with coefficients (3.30).

We mention that a different approach could have been adopted in this case. Indeed we could have required that a four-dimensional subalgebra of the 'free' symmetry algebra (3.13) should survive, as it does in the continuous case for $g = x^{-2}$. This would be the algebra $\{P_0, D, K, W\}$. This algebra could then be inserted into equation (3.22) and that equation solved for the potential $g(x, T_x)$. The result for $g(x, T_x)$ would then be (3.29).

3.2. The second-order difference equation $\Delta_{xx} u = 0$

The ordinary differential equation $u_{xx} = 0$ is invariant under the Lie group $SL(3, \mathbb{R})$, acting as the group of projective transformations of the (x, u)-plane. The six-dimensional subgroup of affine transformations acts linearly and globally. The remaining two one-parameter subgroups act locally and nonlinearly in both x and u. The symmetry group in itself is of no particular use (since we know the general solution $u = ax + b$ anyway). However, Lie point symmetries survive under point transformations. Hence, any ordinary differential equation, linearizable by a point transformation, will have an $sl(3, \mathbb{R})$ symmetry algebra. This is a very useful criterion of linearizability.

We expect the same to be true for the difference equation

$$\Delta_{xx} u = 0 \,. \tag{3.31}$$

The linear formalism and the corresponding ansatz (2.21) are clearly insufficient in this case. We start from the more general ansatz of (2.16), namely that the discrete evolutionary vector field has the form

$$X_e = [\xi(x, T_x^a u) T_x^b \Delta_x u + \phi(x, T_x^c u)]\partial_u = Q\partial_u \,. \tag{3.32}$$

The determining equations are obtained by requiring that we have

$$\Delta_{xx}^T Q|_{\Delta_{xx} u = 0} = 0 \,. \tag{3.33}$$

We expand ξ and ϕ in Laurent series in u and rewrite equation (3.33) as

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} \{ & \Delta_{xx} \xi_j(x, T_x) T_x^2 [u^j \Delta_x u] + 2\Delta_x \xi_j(x, T_x) T_x [(\Delta_x u^j) T_x \Delta_x u] \\ & + \xi_j(x, T_x) [(\Delta_{xx} u^j) T_x^2 \Delta_x u] + \Delta_{xx} \phi_j(x, T_x) T_x^2 u^j \\ & + 2\Delta_x \phi_j(x, T_x) T_x (\Delta_x u^j) + \phi_j(x, T_x)(\Delta_{xx} u^j)\}|_{\Delta_{xx} u = 0} = 0 \end{aligned} \tag{3.34}$$

where ξ_j, ϕ_j are functions of x and operators in T_x. For $\Delta_{xx}u = 0$ we have

$$\begin{aligned}
&\Delta_x u^j = \Delta_x u \sum_{k=1}^{j} u^{k-1}(Tu^{j-k}) \qquad j \geqslant 1 \\
&\Delta_x u^j = 0 \qquad j = 0 \\
&\Delta_x u^j = -\Delta_x u \sum_{k=1}^{-j} u^{k-j-1}(Tu^{-k}) \qquad j < 0
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
&\Delta_{xx}u^j = (\Delta_x u)^2 \sum_{k=1}^{j} \left\{ \sum_{l=1}^{k-1} u^{l-1}(Tu^{k-l-1}(T^2u^{j-k}) + u^{k-1} \sum_{l=1}^{j-k} (Tu^{l-1})(T^2u^{j-k-l}) \right\} \\
&\qquad j > 1 \\
&\qquad \Delta_{xx}u^j = 0 \qquad j = 0, 1 \\
&\Delta_{xx}u^j = (\Delta_x u)^2 \sum_{k=1}^{-j} \left\{ \sum_{k=1}^{-j+1-k} u^{l+j+k-2}(Tu^{-l}) + u^{k+j-1} \sum_{l=1}^{k} (Tu^{l-k-1})(T^2u^{-l}) \right\} \qquad j < 0\,.
\end{aligned} \tag{3.36}$$

To derive (3.34)–(3.36) we have taken into account that for $\Delta_{xx}u = 0$, we have $T_x^n \Delta_x u = \Delta_x u$ for any n. Using equations (3.35) and (3.36), we can rewrite (3.34) as a Laurent series in $u(x)$ and $\Delta_x u$, with coefficients depending on the operators ξ_j and ϕ_j, and their first and second x-variations. By equating to zero all coefficients of different powers of u and $\Delta_x u$ we get the following determining equations:

$$\begin{aligned}
&\xi_j(x, T_x) = 0 \qquad \forall j < 0 \quad \text{and} \quad j \geqslant 2 \\
&\phi_j(x, T_x) = 0 \qquad j < 0 \quad \text{and} \quad j \geqslant 3 \\
&\Delta_{xx}\phi_j = 0 \qquad j = 0, 1, 2 \\
&\Delta_{xx}\xi_0 + 2\Delta_x\phi_1 = 0 \\
&\Delta_{xx}\xi_1 T_x^2 + 2\Delta_x\phi_2 T_x(T_x + 1) = 0 \\
&\Delta_x\xi_1 + 2\phi_2 = 0\,.
\end{aligned} \tag{3.37}$$

Solving equation (3.37), we find that the symmetry algebra is characterized by four constants and four functions of T_x and σ_x and is given by eight vector fields which in the evolutionary formalism are given by

$$\begin{aligned}
&X_{1e} = \Delta_x u \partial_u \\
&X_{2e} = x\Delta_x u \partial_u \\
&X_{3e} = \phi_{10}(T_x, \sigma_x) u \partial_u \\
&X_{4e} = [x\phi_{11}(\sigma_x, T_x)u - x^{(2)}\phi_{11}(\sigma_x, T_x)\Delta_x u]\partial_u \\
&X_{5e} = \partial_u \\
&X_{6e} = x\partial_u \\
&X_{7e} = \xi_{10}(T_x, \sigma_x) u \Delta_x u \partial_u \\
&X_{8e} = \{\phi_{20}(\sigma_x, T_x)u^2 - x\phi_{20}(\sigma_x, T_x)u\Delta_x u\}\partial_u\,.
\end{aligned} \tag{3.38}$$

In order to obtain the $sl(3, \mathbb{R})$ algebra, we choose the remaining functions of σ_x and T_x to be

$$\phi_{10} = 1 \qquad \phi_{11} = T_x^{-1} \qquad \xi_{10} = 1 \qquad \phi_{20} = 1\,. \tag{3.39}$$

Thus, as in the case of the discrete heat equation, we obtain a symmetry algebra, isomorphic to the one that exists in the continuous limit.

Had we made the 'linear' ansatz

$$X_e = (\xi(x)\Delta_x u + f(x)u)\partial_u = Q\partial_u \tag{3.40}$$

or simply used the formalism of commuting operators, we would have only obtained a four-dimensional subalgebra, namely

$$\{X_{1e}, X_{2e}, X_{3e}, X_{4e}\}\,. \tag{3.41}$$

The meaning of the $sl(3,\mathbb{R})$ symmetry algebra in the discrete case is the same as in the continuous one: the corresponding transformations will take straight lines into straight lines. The situation here is so simple, because in this case the discrete equation has the same solution set as its continuous limit.

3.3. Symmetries of the continuous–discrete heat equation

In the present approach, results corresponding to differential-difference equations can be obtained by taking limits in which one, or more, of the spacings go to zero. Thus, e.g. for continuous time we have

$$\sigma_t \to 0 \qquad T_t \to 1 \qquad \Delta_t \to \frac{\partial}{\partial t}\,. \tag{3.42}$$

Let us consider the heat equation in this limit:

$$u_t - \Delta_{xx}u + g(x, T_x)u = 0\,. \tag{3.43}$$

For the free equation we have $g = 0$ and the limit of (3.13) gives

$$\begin{aligned}
&P_0 = u_t\partial_u \qquad P_1 = (\Delta_x u)\partial_u \qquad W = u\partial_u\\
&B = (2t\,\Delta_x u + xT_x^{-1}u + \tfrac{1}{2}\sigma_x T_x^{-1}u)\partial_u\\
&D = [2tu_t + xT_x^{-1}\Delta_x u + (1 - \tfrac{1}{2}T_x^{-1})u]\partial_u\\
&K = \{t^2u_t + txT_x^{-1}\Delta_x u + [\tfrac{1}{4}x^2T_x^{-2} + t(1 - \tfrac{1}{2}T_x^{-1}) - \tfrac{1}{16}\sigma_x^2T_x^{-2}]u\}\partial_u\,.
\end{aligned} \tag{3.44}$$

The symmetry algebras in the discrete–discrete, discrete–continuous and continuous–continuous cases are all mutually isomorphic, though they are all realized differently.

Similarly, we could take the limits for the potentials (3.23) and (3.29) that reduce to $g = k^2x^2$ and $g = 1/x^2$, respectively.

3.4. Symmetries of a nonlinear ordinary difference equation

Let us consider the equation

$$\Delta_{xx}u + u^2 = 0\,. \tag{3.45}$$

In the continuous limit we have

$$u_{xx} + u^2 = 0 \tag{3.46}$$

and this equation is invariant under a two-dimensional symmetry group, generated by translations and dilations

$$P = \partial_x \qquad D = x\partial_x - 2u\partial_u\,. \tag{3.47}$$

The corresponding evolutionary vector fields are

$$P_e = u_x \partial_u \qquad D_e = (xu_x + 2u)\partial_u \,. \tag{3.48}$$

In order to avoid using the complete formalism for nonlinear difference equations, let us make a simplified ansatz, suggested by the continuous limit. Thus, we put

$$Q = \xi(x, T)(\Delta_x u) + \phi(T^a u)\,. \tag{3.49}$$

The determining equations are given by

$$\mathrm{pr}\, X_e(\Delta_{xx} u + u^2)|_{\Delta_{xx} u = -u^2} = 0$$

i.e.

$$Q^{xx} + 2uQ|_{\Delta_{xx} u = -u^2} = 0\,. \tag{3.50}$$

We have

$$\Delta_{xx}^T \xi = \Delta_{xx} \xi \tag{3.51}$$

(since ξ by assumption (3.49) does not depend on u). For ϕ we make use of a Taylor expansion to obtain

$$\begin{aligned}\Delta_{xx}^T \phi = \sum_a \Bigg\{ & T^a (\Delta_x u)^2 \sum_{k=2}^{\infty} \frac{1}{k!} \frac{\partial^k \phi}{\partial (T^a u^k)} (2^k - 2)[(T-1)T^a u]^{k-2} \\ & + (T^a \Delta_{xx} u) \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \phi}{\partial (T^a u^k)} \sum_{m=1}^{k} \binom{k}{m} [(T^2 - 2T + 1)T^a u]^{m-1} \\ & \times [2T^a (T-1)u]^{k-m} \Bigg\}.\end{aligned} \tag{3.52}$$

Substituting into (3.51) and requiring that the coefficient of $(\Delta_x u)^2$ must vanish we find

$$\frac{\partial^k \phi}{\partial (T^a u^2)} = 0 \qquad k \geqslant 2$$

i.e.

$$\phi = A + B(T)u\,. \tag{3.53}$$

The coefficient of $(\Delta_x u)$ must also vanish. This implies

$$\xi = C(T) + xD(T)\,. \tag{3.54}$$

We expand all functions of the shift operator T into series, e.g.

$$C(T) = \sum_{k=0}^{\infty} \gamma_k (T-1)^k \tag{3.55}$$

(inspired by the continuous limit $T \to 1$) and obtain $A = 0$, $D(T) = 0$

$$\begin{aligned} C(T) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (T-1)^k = (T-1)^{-1} \ln T \\ B(T) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (T-1)^k = \ln T\,. \end{aligned} \tag{3.56}$$

The corresponding evolutionary fields

$$\begin{aligned}\widehat{X}_e &= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (T-1)^k \Delta_x u\right]\partial_u \\ \widehat{Y}_e &= \left[\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (T-1)^k u\right]\partial_u\end{aligned} \tag{3.57}$$

satisfy

$$\widehat{X}_e \to P_e \qquad \widehat{Y}_e \to 0 \tag{3.58}$$

in the continuous limit.

Thus, translational invariance is represented by the operator $\widehat{X}_e$, but dilational invariance is not caught by the simplified ansatz (3.49). On the other hand, a new symmetry $\widehat{Y}_e$, occurring only in the discrete case, makes its appearance.

4. Conclusions

The main result presented in this paper is a constructive general formalism for calculating symmetries of difference equations. It is applicable to arbitrary difference equations for a set of functions $u_\alpha(x)$ $(1 \leqslant \alpha \leqslant q)$ of p independent variables x_i. The dependent and independent variables are viewed as continuous, but the equations involve finite differences $\Delta_x u$, rather than derivatives.

The proposed formalism is a nontrivial extension of the formalism of evolutionary vector fields used for differential equations. For difference equations it turns out to be essential to incorporate a certain class of generalized symmetries, in order to allow for discrete variations of the independent variables. In the continuous limit these symmetries reduce to point ones (since all operators T satisfy $T \to 1$ in the continuous limit $\sigma \to 0$).

The method of obtaining the determining equations is not an *ad hoc* one and is not just based on an analogy with the continuous case. Indeed, it is easy to show that the condition

$$\operatorname{pr} X_e E|_{E=0} = 0 \tag{4.1}$$

is equivalent to the requirement that

$$\mathrm{e}^{\lambda X_e} u(x) = \tilde{u}(x, \lambda) \tag{4.2}$$

is a solution, whenever $u(x)$ is one (in the discrete, just as in the continuous case). Moreover, if (4.2) gives a solution for $\lambda \ll 1$, it does so also for λ finite (just as in the continuous case).

When deriving the determining equations, it is essential to distinguish between shift operators T_i and difference operators Δ_{x_i}. Obviously, they are related by (1.3), involving the spacing σ_i. In deriving determining equations from expression (4.1), we set equal to zero the coefficients of different expressions in $(\Delta_{x_i} u)$, $(\Delta_{x_i} \Delta_{x_k} u)$, etc. The functions ξ_i and ϕ in the vector field (2.16) are assumed not to depend on the spacing σ_i, but only on x and u, where, however, u can be shifted arbitrarily $(T_{x_i}^{\alpha_i} u)$. We note that expressions like $(T-1)$ tend to zero for $\sigma \to 0$, whereas $\Delta_x u$ tends to the derivative $\partial u / \partial x$ (thus $(T-1)u$ cannot 'simulate' a variation).

Once a Lie algebra of vector fields is established and a basis is chosen, we can take linear combinations of elements with coefficients that depend on the spacings σ_i. This was done in the case of the discrete heat equation. Indeed, in (3.13), $W = u\partial_u$ is a symmetry operator.

The terms proportional to σ_x in B and σ_x^2 in K involve precisely such a spacing-dependent linear combination.

The examples treated in this paper have brought out several features.

First of all, the present approach gives a larger class of symmetries than those obtained when requiring that the discrete variables vary only on a given fixed lattice [14–18]. As a matter of fact, for linear equations we have always obtained a Lie algebra isomorphic to that obtained in the continuous limit.

The second conclusion is that the formalism needs further improvements. While we can handle symmetries of linear equations in complete generality, difficulties arise for nonlinear ones. In order to obtain a manageable set of determining equations, we were forced to impose *a priori* restrictions on the form of the vector fields. The symmetry algebra thus obtained will, in general, be a subalgebra of the entire algebra (though still larger than that obtained by other methods).

A paper addressing these problems is in preparation.

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